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## **Prolongation structure and Lax representation of the Boomeron equation**

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### **ABSTRACT**

An Estabrook-Wahlquist prolongation structure and a Lax representation for the Boomeron equation introduced by Calogero and Degasperis is determined.

### **1. INTRODUCTION**

In the present paper we report a number of results for the boomeron nonlinear evolution equation, that has been introduced not very long ago by Calogero and Degasperis and shown to be solvable through the inverse scattering technique associated with the matrix Schrödinger equation [1, 2, 3]. Wahlquist and Estabrook introduced a method, which they call prolongation, to study nonlinear equations [4]. This method has been applied with success to an increasing number of nonlinear partial differential equations solvable through inverse scattering and admitting soliton-like solutions.

Lax in his very fundamental paper [6] was able to write the Korteweg-de Vries equation as a statement concerning the unitary equivalence of operators and clarified in this way several properties of the KdV. This aspect is now appropriately termed the Lax approach and studied in connection with a large number of nonlinear equations.

Here we will determine a prolongation structure for the Boomeron equation and then derive a Lax representation for this equation by means of this prolongation structure.

## 2. THE BOOMERON EQUATION

For our convenience the Boomeron equation is written in the form

$$(2.1) \quad W_{tx} = [\alpha, W_x] + \{W_{xx}, \beta\} + [W_x, [W, \beta]],$$

where  $\alpha, \beta$  are given  $2 \times 2$ -matrices [3].  $W$  is the unknown  $(2 \times 2)$ -matrix depending on  $x$  and  $t$  and the subscripts indicate partial differentiation.  $[A, B]$  and  $\{A, B\}$  denote the commutator and anticommutator of the matrices  $A$  and  $B$  respectively. The matrix  $W$  is assumed to be hermitian and  $\alpha$  and  $\beta$  of the form

$$(2.2) \quad \alpha = \alpha_1 \sigma_1 + \alpha_2 \sigma_2 + \alpha_3 \sigma_3, \quad \beta = \beta_1 \sigma_1 + \beta_2 \sigma_2 + \beta_3 \sigma_3.$$

Here  $\sigma_n$  ( $n = 1, 2, 3$ ) are the Pauli matrices given by

$$(2.3) \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Asymptotic boundary conditions for equation (2.1) are expressed by

$$(2.4) \quad W(\infty, t) = 0, \quad W_x(\pm \infty, t) = 0.$$

Calogero and Degasperis obtained equation (2.1) by applying a technique based on generalized wronskian relations and usually write this equation in a different but equivalent form.

## 3. DETERMINATION OF A PROLONGATION STRUCTURE

Prolongation may be carried out using the machinery of exterior differential systems [4, 5] or of jet bundles (8). Our analysis can be described without using these formalisms. Starting point of the discussion here is the following overdetermined system of linear differential equations

$$(3.1) \quad \begin{pmatrix} y \\ z \end{pmatrix}_x = \left( \begin{array}{c|c} A & I \\ \hline C & -A \end{array} \right) \begin{pmatrix} y \\ z \end{pmatrix}, \quad \begin{pmatrix} y \\ z \end{pmatrix}_t = \left( \begin{array}{c|c} a & 2b \\ \hline c & -a \end{array} \right) \begin{pmatrix} y \\ z \end{pmatrix},$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \text{ and } z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

are 2-vectors and  $A, C, a, b, c$  ( $2 \times 2$ )-matrices depending on  $x$  and  $t$  and to be specified later.

From the first equation of (3.1) we obtain

$$y_{xx} = (Ay + z)_x = A_x y + A(Ay + z) + Cy - Az = (A_x + A^2 + C)y.$$

Making the choice  $C = -A^2 - k^2$  and taking  $A_x = Q$  we obtain the following eigenvalue problem

$$(3.2) \quad y_{xx} = (Q - k^2)y,$$

the well-known matrix Schrödinger equation.

The compatibility conditions for the system (3.1) are expressed by the following set of equations

$$(3.3.a) \quad A_t - a_x + [A, a] + c + 2b(A^2 + k^2) = 0$$

$$(3.3.b) \quad -b_x + \{A, b\} - a = 0$$

$$(3.3.c) \quad -(A^2)_t - c_x - \{a, A^2 + k^2\} - \{c, A\} = 0$$

$$(3.3.d) \quad -A_t + a_x - 2(A^2 + k^2)b + [A, a] - c = 0$$

We simplify this set of equations by taking for  $b$  a constant matrix. In this case it follows from (3.3.b) that

$$(3.4) \quad [a, A] = [\{A, b\}, A] = [b, A^2]$$

and the system (3.3) reduces to

$$(3.5.a) \quad A_t - \{A_x, b\} + \{A^2, b\} + 2k^2b + c = 0$$

$$(3.5.b) \quad -(A^2)_t - c_x - \{\{A, b\}, A^2 + k^2\} - \{c, A\} = 0$$

$$(3.5.c) \quad a = \{A, b\}.$$

From equation (3.5.a) the matrix  $c$  is determined as an expression in  $A$  and  $b$  and in (3.5.c) the matrix  $a$  is written as the anticommutator of  $A$  and  $b$ . Now substitution of these expressions in (3.5.b) leads to a single equation for the matrix  $A$ , which after a small calculation becomes

$$(3.6) \quad A_{tx} - \{A_{xx}, b\} + \{(A^2)_x, b\} - \{\{A_x, b\}, A\} = 0.$$

Another simple calculation shows

$$(3.7) \quad \{(A^2)_x, b\} - \{\{A_x, b\}, A\} = [A_x, [A, b]].$$

Consequently equation (3.6) is equivalent to

$$(3.8) \quad A_{tx} = \{A_{xx}, b\} - [A_x, [A, b]].$$

Setting

$$(3.9) \quad A = -W + \gamma \quad b = \beta \text{ and } [\gamma, \beta] = \alpha,$$

where  $\alpha, \beta, \gamma$  are constant  $(2 \times 2)$ -matrices this equation (3.8) becomes

$$(3.10) \quad W_{tx} = [\alpha, W_x] + \{W_{xx}, \beta\} + [W_x, [W, \beta]],$$

which is the Boomeron equation (2.1). So we have found a prolongation structure for the Boomeron equation in case the matrix  $\alpha$  equals the commutator of  $\beta$  with another constant matrix.

Summarizing, the prolongation is given by (3.1), where  $A, b$  are given by (3.9),  $a$  by (3.5.c) and the more complicated expression for  $c$  by (3.5.a).

#### 4. A LAX REPRESENTATION

From the prolongation described in section 3 it is possible to derive a Lax representation in the following way. As a consequence of (3.1), (3.5.c) we have

$$y_x = Ay + z$$

and

$$y_t = \{A, b\}y + 2bz$$

which implies that

$$y_t = \{A, b\}y + 2b(y_x - Ay) = [A, b]y + 2by_x$$

and by substitution of (3.9) we obtain

$$(4.1) \quad y_t = (-[W, \beta] + \alpha)y + 2\beta y_x.$$

Now define operators  $L$  and  $B$  by

$$(4.2) \quad L = D^2 + W_x, \quad B = -[W, \beta] + \alpha + 2\beta D,$$

where  $D$  denotes the differential operator with respect to  $x$ . Then (3.2) with  $Q = -W_x$  and (4.1) can be written in the form

$$(4.3) \quad Ly = -k^2 y, \quad y_t = By.$$

The operators in (4.2) together with the Boomeron equation in the form (3.10) suggest that the Boomeron equation (3.10) may be written in the Lax representation

$$(4.4) \quad L_t = [B, L].$$

Indeed, some computations yield

$$\begin{aligned} BLy &= (-[W, \beta] + \alpha + 2\beta D)(D^2 + W_x)y = \\ &= 2\beta y_{xxx} + (-[W, \beta] + \alpha)y_{xx} + 2\beta W_x y_x + (-[W, \beta]W_x + \alpha W_x + 2\beta W_{xx})y \end{aligned}$$

and

$$\begin{aligned} LBy &= 2\beta y_{xxx} + (-[W, \beta] + \alpha)y_{xx} + 2(-[W_x, \beta] + W_x \beta)y_x + \\ &+ (-W_x[W, \beta] + W_x \alpha - [W_{xx}, \beta])y \end{aligned}$$

which implies that

$$[B, L]y = (BL - LB)y = ([W_x, [W, \beta]] + [\alpha, W_x] + \{W_{xx}, \beta\})y$$

and because  $L_t = W_{xt}$  we conclude that  $L_t y = [B, L]y$  is equivalent to (3.10).

Note that the Lax representation (4.4) holds for arbitrary matrices  $\alpha$  and  $\beta$ . However the operator  $B$  is antisymmetric in case  $\beta$  is hermitian and  $\alpha$  anti-hermitian, i.e.  $\beta_n$  real and  $\alpha_n$  pure imaginary ( $n = 1, 2, 3$ ) in the expressions for  $\alpha$  and  $\beta$  in (2.2). So in this special case the Boomeron equation may be interpreted as the isospectral deformation of the symmetric Schrödinger operator  $L$  with respect to  $B$ .

Following Lax [7] the evolution of scattering data is easily determined using this isospectral deformation. We will illustrate this for the reflection coefficient. The (matrix) reflection coefficient  $R$  and transmission coefficient  $T$  are characterized by the asymptotic boundary conditions

$$(4.5) \quad \begin{cases} \psi(x, k) \simeq I \exp(-ikx) + R(k) \exp(ikx) & (x \rightarrow +\infty) \\ \simeq T(k) \exp(-ikx) & (x \rightarrow -\infty) \end{cases}$$

for the matrix Schrödinger equation [3]

$$(4.6) \quad \phi_{xx} = (Q - k^2)\phi \quad Q = -W_x,$$

where it is assumed that the “potential”  $Q$  vanishes asymptotically sufficiently fast and  $I$  is the unit matrix.

Suppose  $\phi(x, t_0)$  is normalized as in (4.5); since by assumption  $W$  is very small it follows from (4.1) that  $\phi$  evolves according to

$$(4.7) \quad \phi_t = 2\beta\phi_x + \alpha\phi$$

and from (4.6) it follows that for large  $|x|$   $\phi$  is a linear combination of  $\exp(ikx)$  and  $\exp(-ikx)$ , i.e.

$$(4.8) \quad \phi(x, t, k) \simeq A(t, k) \exp(-ikx) + B(t, k) \exp(ikx) \quad (|x| \rightarrow \infty).$$

The evolution of  $\phi$  given in (4.7) together with the asymptotic description (4.8) imply that the coefficients  $A$  and  $B$  evolve according to

$$(4.9) \quad \begin{cases} A_t = (-2ik\beta + \alpha)A \\ B_t = (2ik\beta + \alpha)B. \end{cases}$$

Solutions of these equations are

$$A(t, k) = (\exp(t - t_0)(-2ik\beta + \alpha))A(t_0, k)$$

$$B(t, k) = (\exp(t - t_0)(2ik\beta + \alpha))B(t_0, k)$$

with  $A(t_0, k) = I$  and  $B(t_0, k) = R(t_0, k)$ .

Consequently

$$(4.10) \quad \begin{cases} \phi(x, t, k) \simeq (\exp(t - t_0)(-2ik\beta + \alpha)) \exp(-ikx) + \\ + (\exp(t - t_0)(2ik\beta + \alpha))R(t_0, k) \exp(ikx) & (x \rightarrow +\infty). \end{cases}$$

To bring  $\phi$  into the form (4.5) we have to multiply  $\phi$  to the right by the matrix  $\exp(t - t_0)(2ik\beta - \alpha)$ . Then we have

$$\begin{aligned} \psi(x, t, k) &\simeq I \exp(-ikx) + (\exp(t - t_0)(2ik\beta + \alpha))R(t_0, k) \times \\ &\times (\exp(t - t_0)(2ik\beta - \alpha)) \exp(ikx) \quad (x \rightarrow +\infty). \end{aligned}$$

This reveals

$$R(t, k) = \exp(t - t_0)(2ik\beta + \alpha)R(t_0, k) \exp(t - t_0)(2ik\beta - \alpha)$$

and

$$\begin{aligned} R_t(t, k) &= (2ik\beta + \alpha)R(t, k) + R(t, k)(2ik\beta + \alpha) = \\ &= 2ik\{\beta, R(t, k)\} + [\alpha, R(t, k)] \end{aligned}$$

corresponding to the relations 8a in ref. 1 and 4.13 in ref. 3 derived by Calogero and Degasperis.

## 5. CONCLUSION AND DISCUSSION

In the Estabrook-Wahlquist prolongation procedure for a given partial differential equation one starts with a differential ideal of differential forms representing the equation and then prolonging this ideal by introduction of a Lie algebra valued one-form, a connection form. We note that in this approach for hermitian  $\gamma$  the Lie algebra associated with the prolongation obtained here consists of matrices  $M$  having trace zero and which are of the form

$$M = \left( \begin{array}{c|c} A & B \\ \hline C & -A^* \end{array} \right),$$

where the  $(2 \times 2)$ -matrix  $A$  is arbitrary and  $B$  and  $C$  are hermitian  $(2 \times 2)$ -matrices. This Lie algebra may be characterized as the Lie algebra of  $(4 \times 4)$ -matrices  $M$  of trace zero satisfying

$$\left\langle M \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} y \\ z \end{pmatrix}, M \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle = 0$$

for all  $\begin{pmatrix} y \\ z \end{pmatrix}$  and  $\begin{pmatrix} u \\ v \end{pmatrix}$ . Here  $\langle \ , \ \rangle$  is the sesqui-linear form defined by

$$\left\langle \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle = y\bar{v} - z\bar{u}$$

in which the bar denotes complex conjugation. So the Lie group underlying the prolongation structure consists of endomorphisms preserving this sesqui-linear form and having determinant equal to 1.

A close inspection of the derivation of the Lax representation in section 4 shows that the representation (4.4) remains valid in case the matrices  $\alpha$  and  $\beta$  depend on  $t$ . The corresponding partial differential equation is briefly discussed by Calogero and Degasperis in ref. 3, section 8.

A particular interesting problem would be to obtain a Bäcklund transformation from the prolongation structure described in section 3. Another significant investigation would be to clarify the appearance of the Lie algebra in the prolongation described before. As well as a study of the Lax representation in connection with conservation laws.

We hope to report with regard to these in future work.

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